”SIMION STOILOW”
INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY

DOCTORAL THESIS

ABSTRACT

APLICATIONS OF THE HOMOGENIZATION METHOD IN DIFUSION PROBLEMS

Ph.D. Adivsor,

Prof. Univ. Dr. Horia ENE

Ph.D Student,

Nicolae - Iulian ŢENŢEA

Bucharest, 2014

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Introduction

The aim of this paper is applying the homogenization method to some elasticity and thermoelasticity problems, respectively, which are stated in a domain composed of two components. There are studied several cases according the conditions on the interface between the two components of the domain and the structure of the medium that occupies it.

Regarding the considered domain, it is an open subset Ω of $\mathbb{R}^N$ $(N \geqslant 2)$, whose boundary $\partial\Omega$ is Lipschitz continuous, and it is described at the begining of Chapter 1. The domain is occupied by a medium with periodic structure, composed by two components, one of them being connected and the other disconnected. The interface between the two components has the necesary properties for formulating the problems studied in this paper. More exactly, we consider $Y = (0,1)^N$ the unit cube of $\mathbb{R}^N$ and we assume that Y_2 is a subset of Y such that $\bar{Y}_2 \subset Y$ and it's boundary Γ is also Lipschitz continuous. We define $Y_1 = Y \setminus \bar{Y}_2$ and one can observe that repeating Y by periodicity, the union of all $\bar{Y}_1$ forms a connected domain in $\mathbb{R}^N$ which will be denoted by $\mathbb{R}^N_1$. We also consider $\mathbb{R}^N_2 = \mathbb{R}^N \setminus \mathbb{R}^N_1$.

in what follows, the parameter $\varepsilon \in (0,1)$ represents the dimension of the periodicity cell and it will take it's values in a sequence of real numbers, which, in the homogenization process, will converge to zero. For each $k \in \mathbb{Z}^N$ we set $Y^k = k + Y$ and $Y^k_\alpha = k + Y_\alpha$ where $\alpha \in \{1,2\}$. We define, for each ε, $\mathbb{Z}_\varepsilon = \left\{k \in \mathbb{Z}^N : \ \varepsilon \bar{Y}^k_2 \subset \Omega\right\}$, and we introduce the sets

$$\Omega^\varepsilon_2 = \bigcup_{k \in \mathbb{Z}_\varepsilon} \left(\varepsilon Y^k_2\right) \quad \text{and} \quad \Omega^\varepsilon_1 = \Omega \setminus \bar{\Omega}^\varepsilon_2,$$

which represent the two components of domain Ω. The boundary of Ω^ε_2 will be denoted by Γ_ε and n will represent the normal vector on Γ_ε, exterior to Ω^ε_1. The interface Γ_ε represents in fact a thin layer which surrounds the particles and it is modeled as a surface.

For homogenization of the problems considered in this paper we use the periodic *unfolding* method introduced by Ciorănescu, Damlamian and Griso in [5], which was later extended to perforated domains by Cioranescu, Damlamian, Donato, Griso and Zaki in [9] and [7]. In [12] Donato *et al.* use the perodic unfolding method for a two component domain similar to the one considered in this paper. Later, Doanto and Yang use in [15] a time depending unfolding operator for a wave problem, in a perforated domain. In [38], Yang defines two time depending unfolding operators in a domain similar to the one considered in this paper

Regarding the paper's structure, it is divided in four chapters. In the first part of Chapter 1 we describe the domain and we continue with the study of an elasticity problem. We consider that we have a double porosity medium, more exactly the elasticity of the disconnected component is of order ε^2. On the interface between the two components of the medium, we have a jump of the displacement vector, proportional with the normal component of the stress tensor which is supposed continuous. After obtaining the variational formulation of the problem and a priori estimates we prove some convergence results and we get the coupled homogenized problem by passing to limit as $\varepsilon \to 0$. After that we decouple the limit problem by introducing the homogenized coefficients and the solutions of cell problems. Although the elasticity tensor of order ε^2 does not make part of the homogenized tensor, it make its presence felt in the solution of the limit problem through the cell problems formulated in Y_2.

The next three chapters are dedicated to some thermoelasticity problems with zero initial conditions. More exactly, we study the difusion of the temperature in an elastic medium which occupies the domain Ω defined in Chapter 1. Several cases are approached according to the conditions considered on the interface between the two components of the domain and to the form of the elasticity tensor in the disconnected component. As in Chapter 2, we obtain the homogenized problem, coresponding to each thermoelasticity model that is proposed, and one can see a combination of the homogenization results of an elasticity problem with the results of a diffusion one.

More exactly, in Chapter 2 we study the clasic thermoelastic model and we add jumps conditions of both displacements and temperatures, on the interface between the components of the medium. The homogenized problem obtained in this case is similar to the initial thermoelastic one, the differences being represented by the presence of some coupling terms between limits u^1 and u^2, respectively θ^1 and θ^2. Also, the tensors which describe the disconnected component do not appear in the homogenized problem.

In Chapter 3 we analize the same problem but we consider again that the elasticity of the disconnected component is of order ε^2. Also, the temperature-displacement tensor and the density are of order ε in the disconnected component. This time, the tensors which describe the disconnected component appear in the homogenized problem, more exactly they make their presence felt in the homogenized equation of the temperature, resulted by passing to limit on the disconnected component. As in Chapter 1, they are also part of solution of the limit problem through the cell problems stated in Y_2. Moreover, as in Chapter 1, one can see that the coupling term of limits u^1 and u^2 which describe the displacements, does not exists in this case.

The last model proposed in this paper is studied in Chapter 4. This time we consider that only the displacements have a jump on the interface between the two components of the medium. The disconnected component has also the elasticity of order ε^2 and its density and the temperature-displacement tensor are of order ε. The difference between this model and the one studied in Chapter 3, consists, as we expected, in the absence of the coupling term of the limits which describe the temperatures.

The models proposed in this thesis have not been treated before, thus the results exposed here are original, as they are obtained from my own research activity.

1 Homogenization of an elastic double porosity medium with imperfect interface

In this Chapter we study an elasticity problem stated in a double porosity medium which occpupies the domain Ω. On the interface Γ_ε we consider a jump of the displacements, proportional with the normal component of the stress tensor which is supposed continuous. More exactly, we have the problem

$$\begin{cases} -\dfrac{\partial \sigma_{ij}^{\alpha\varepsilon}}{\partial x_j} = g_i \quad \text{in } \Omega_\alpha^\varepsilon, \quad \alpha \in \{1,2\}, \\ \sigma_{ij}^{1\varepsilon} n_j = \sigma_{ij}^{2\varepsilon} n_j = \varepsilon h_\varepsilon (u_i^{2\varepsilon} - u_i^{1\varepsilon}) \quad \text{on } \Gamma_\varepsilon, \\ u^{1\varepsilon} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $h_\varepsilon(x) = h(x/\varepsilon)$ represents the jump factor $\sigma_{ij}^{\alpha\varepsilon} = a_{ijkh}^{\alpha\varepsilon} e_{kh}(u^{\alpha\varepsilon})$ are the components of the stress tensors. The functions $e_{kh}(u^{\alpha\varepsilon}) = \frac{1}{2}\left(\frac{\partial u_k^{\alpha\varepsilon}}{\partial x_h} + \frac{\partial u_h^{\alpha\varepsilon}}{\partial x_k}\right)$ represent the components of the deformation tensor and $a_{ijkh}^{\varepsilon\alpha}$ are the components of the elasticity tensors defined by:

$$A^{1\varepsilon}(x) = A^1(x/\varepsilon) \text{ and } A^{2\varepsilon}(x) = \varepsilon^2 A^2(x/\varepsilon). \tag{1.2}$$

We consider that h and the components a_{ijkh}^α of the symmetric and positive definite tensors A^α, are smooth, Y- periodic adn bounded functions an we also consider that $h(y) > 0$ on Γ. We introduce the space $V_\varepsilon = \{v \in H^1(\Omega_1^\varepsilon),\ v = 0 \text{ pe } \partial\Omega\}$ endowed with the L^2 norm of the gradients and the Hilbert space

$$H_\varepsilon = V_\varepsilon^N \times H^1(\Omega_2^\varepsilon)^N \tag{1.3}$$

endowed with the scalar product

$$(u,v)_{H_\varepsilon} = \int_{\Omega_1^\varepsilon} \nabla u_i^1 \nabla v_i^1 + \varepsilon^2 \int_{\Omega_2^\varepsilon} \nabla u_i^2 \nabla v_i^2 + \varepsilon \int_{\Gamma_\varepsilon} (u_i^2 - u_i^1)(v_i^2 - v_i^1), \tag{1.4}$$

where the elements of H_ε are denoted $u = (u^1, u^2)$. The variational formulation of problem (1.1) is:

Find $u^\varepsilon \in H_\varepsilon$ such that

$$a(u^\varepsilon, v) = \sum_{\alpha=1,2} \int_{\Omega_\alpha^\varepsilon} a_{ijkh}^{\alpha\varepsilon} \frac{\partial u_k^{\alpha\varepsilon}}{\partial x_h} \frac{\partial v_i^\alpha}{\partial x_j} + \varepsilon \int_{\Gamma_\varepsilon} h_\varepsilon (u_i^{2\varepsilon} - u_i^{1\varepsilon})(v_i^2 - v_i^1) = \int_{\Omega_1^\varepsilon} f_i v_i^1 + \int_{\Omega_2^\varepsilon} f_i v_i^2, \quad \forall v \in H_\varepsilon. \tag{1.5}$$

Theorem 1.1. *For any $\varepsilon \in (0,1)$, problem (1.5) has a unique solution $u^\varepsilon \in H_\varepsilon$. Moreover, there exists a constant $C > 0$ independent of ε such that, for $\alpha \in \{1,2\}$ and each $i = 1, \ldots, N$, we have*

$$\|u_i^{\varepsilon\alpha}\|_{L^2(\Omega_\alpha^\varepsilon)} \leqslant C, \quad \|\nabla u_i^{1\varepsilon}\|_{L^2(\Omega_1^\varepsilon)} \leqslant C, \quad \varepsilon \|\nabla u_i^{2\varepsilon}\|_{L^2(\Omega_2^\varepsilon)} \leqslant C, \quad \|u_i^{2\varepsilon} - u_i^{1\varepsilon}\|_{L^2(\Gamma_\varepsilon)} \leqslant C\varepsilon^{-1/2}. \tag{1.6}$$

For a set $D \subset \mathbb{R}^N$ and $v \in L^1(D)$, we denote by $\langle v \rangle_D = \frac{1}{|D|}\int_D v(y)\,dy$ and if v is a funtion definite on $\Omega_\alpha^\varepsilon$, $\alpha \in \{1,2\}$, then $\tilde{v}$ the extension with zero to the entire Ω. Furthermore, we define the spaces:

$$H_{per}^1(Y_\alpha) = \left\{v \in H_{loc}^1(\mathbb{R}_\alpha^N) :\ v \text{ este } Y\text{-periodică}\right\}, \quad \tilde{H}_{per}^1(Y_\alpha) = \left\{v \in H_{per}^1(Y_\alpha) :\ \langle v \rangle_Y = 0\right\},$$

$$V = H_0^1(\Omega)^N \times L^2\left(\Omega; H_{per}^1(Y_1)\right)^N \times L^2\left(\Omega; H^1(Y_2)\right)^N.$$

Using periodic unfolding method we prove some convergence results and we obtain the coupled homogenized problem (in variables x and y).

Theorem 1.2. *If $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ is the solution of problem (1.1), then*

$$\begin{aligned}
&\tilde{u}^{1\varepsilon} \rightharpoonup |Y_1| \cdot u^1 \quad \text{weakly in} \;\; L^2(\Omega)^N, \\
&\tilde{u}^{2\varepsilon} \rightharpoonup |Y_2| \cdot \langle \widehat{u}^2 \rangle_{Y_2} \quad \text{weakly in} \;\; L^2(\Omega)^N, \\
&\mathcal{T}_1^\varepsilon(u^{1\varepsilon}) \longrightarrow u^1 \quad \text{strongly in} \;\; L^2\left(\Omega; H^1(Y_1)\right)^N, \\
&\mathcal{T}_2^\varepsilon(u^{2\varepsilon}) \rightharpoonup \widehat{u}^2 \quad \text{weakly in} \;\; L^2(\Omega; H^1(Y_2))^N, \\
&\mathcal{T}_1^\varepsilon(e_{kh}(u^{1\varepsilon})) \rightharpoonup e_{kh}(u^1) + e^y_{kh}(\widehat{u}^1) \quad \text{weakly in} \;\; L^2(\Omega \times Y_1), \\
&\varepsilon \mathcal{T}_2^\varepsilon(e_{kh}(u^{2\varepsilon})) \rightharpoonup e^y_{kh}(\widehat{u}^2) \quad \text{weakly in} \;\; L^2(\Omega \times Y_2),
\end{aligned} \tag{1.7}$$

where the triplet $(u^1, \widehat{u}^1, \widehat{u}^2) \in V$ with $\langle \widehat{u}^1_i \rangle_\Gamma = 0$ a.e. on Ω, is the unique solution of problem

$$\begin{aligned}
&\int_{\Omega \times Y_1} a^1_{ijkh} \left(\frac{\partial u^1_k}{\partial x_h} + \frac{\partial \widehat{u}^1_k}{\partial y_h} \right) \left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \Phi^1_i}{\partial y_j} \right) + \int_{\Omega \times Y_2} a^2_{ijkh} \frac{\partial \widehat{u}^2_k}{\partial y_h} \frac{\partial \Phi^2_i}{\partial y_j} + \int_{\Omega \times \Gamma} h(\widehat{u}^2_i - u^1_i)(\Phi^2_i - \varphi_i) = \\
&= \int_{\Omega \times Y_1} f_i \varphi_i + \int_{\Omega \times Y_2} f_i \Phi^2_i, \quad \forall (\varphi, \Phi^1, \Phi^2) \in V.
\end{aligned} \tag{1.8}$$

The homogenized problem in Ω is obtained by introducing in (1.8) the expressions of functions $\widehat{u}^1$, respectively $\widehat{u}^2$ and using the homogenized coefficients a^*_{ijlm} formula. More exactly,

$$\widehat{u}^1_k(x,y) = w^{lm}_{1k}(y) \cdot \frac{\partial u^1_l}{\partial x_m}(x) \quad \text{in } \Omega \times Y_1, \tag{1.9}$$

$$\widehat{u}^2_k(x,y) = u^1_k(x) + f_l(x) w^l_{2k}(y) \quad \text{in } \Omega \times Y_2, \tag{1.10}$$

$$a^*_{ijlm} = \int_{Y_1} a^1_{ijlm} + a^1_{ijkh} \frac{\partial w^{lm}_{1k}}{\partial y_h}, \tag{1.11}$$

where for $l, m = 1, \dots N$, $w^{lm}_1 \in \tilde{H}^1_{per}(Y_1)^N$ and $w^l_2 \in H^1(Y_2)^N$ are the unique solutions of cell problems

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(a^1_{ijlm} + a^1_{ijkh} \dfrac{\partial w^{lm}_{1k}}{\partial y_h} \right) = 0 \quad \hat{\imath}\text{n } Y_1 \\[2ex] \left(a^1_{ijlm} + a^1_{ijkh} \dfrac{\partial w^{lm}_{1k}}{\partial y_h} \right) n_j = 0 \quad \text{pe } \Gamma, \end{cases} \qquad \begin{cases} -\dfrac{\partial}{\partial y_j}\left(a^2_{ijkh} \dfrac{\partial w^l_{2k}}{\partial y_h} \right) = \delta_{il} \quad \hat{\imath}\text{n } Y_2 \\[2ex] a^2_{ijkh} \dfrac{\partial w^l_{2k}}{\partial y_h} n_j = h w^l_{2i} \quad \text{pe } \Gamma. \end{cases} \tag{1.12}$$

Theorem 1.3. *If $u^\varepsilon \in H_\varepsilon$ is the solution of problem (1.5), then*

$$\tilde{u}^{1\varepsilon} \rightharpoonup |Y_1| \cdot u^1 \quad \text{weakly in} \;\; L^2(\Omega)^N, \tag{1.13}$$

$$\tilde{u}^{2\varepsilon} \rightharpoonup |Y_2| \cdot u^1 + f_l \cdot q^l \quad \text{weakly in} \;\; L^2(\Omega)^N, \tag{1.14}$$

where u^1 is the unique solution of problem

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a^*_{ijkh} \dfrac{\partial u^1_k}{\partial x_h} \right) = f_i \quad \text{in } \Omega \\[2ex] u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

and the components of q^l are $q^l_i = \displaystyle\int_{Y_2} w^l_{2i}$.

2 The thermoelastic model with jumps in displacements and temperatures, in a two component domain

Starting with this chapter, we will focus on a thermoelasticity problem with zero initial conditions, stated in the domain Ω defined in Chapter 1. We consider two jump factors $h^u_\varepsilon(x) = h^u(x/\varepsilon)$ and $h^\theta_\varepsilon(x) = h^\theta(x/\varepsilon)$, respectively, and the elasticity tensors $A^{\alpha\varepsilon}(x) = A^{\alpha}(x/\varepsilon)$ where h^u, $h^\theta \in L^\infty(\Gamma)$ and the components $a^{\alpha}_{ijkh} \in L^\infty(Y)$ of the symmetric and positive definite tensors A^α smooth, real and Y- periodic functions.

We also introduce the temperature-displacement second order tensors $B^{1\varepsilon}(x) = B^1(x/\varepsilon)$ and $B^{2\varepsilon}(x) = \varepsilon B^2(x/\varepsilon)$ and the thermic conductivity tensors $K^{\alpha\varepsilon}(x) = K^\alpha(x/\varepsilon)$ where B^α and K^α are symmetric, K^α being also positive definite. Their components b^α_{ij} and k^α_{ij}, respectively, are also smooth and Y-periodic functions from $L^\infty(Y)$. Furthermore, T_0 denotes the reference temperature, $\rho^{\alpha\varepsilon}(x) = \rho^\alpha(x/\varepsilon)$ represent the densities of the two mediums, and $c^{\alpha\varepsilon}(x) = c^\alpha(x/\varepsilon)$ is the specific heat for constant deformation of each of the two mediums, the functions $\rho^\alpha, c^\alpha \in L^\infty(Y)$ being considered smooth, Y-periodic and obviously, strictly positive. For $\alpha \in \{1, 2\}$, if $u^{\alpha\varepsilon}$ and$\theta^{\alpha\varepsilon}$ are functions defined on $\Omega^\varepsilon_\alpha$ we introduce the constitutive laws $\sigma^{\alpha\varepsilon}_{ij} = a^{\alpha\varepsilon}_{ijkh} e_{kh}(u^{\alpha\varepsilon}) - b^{\alpha\varepsilon}_{ij}\theta^{\alpha\varepsilon}$.

The problem studied in this chapter is represented by ecuations (2.1)-(2.1) and conditions (2.3)-(2.6):

$$-\frac{\partial \sigma^{\alpha\varepsilon}_{ij}}{\partial x_j} + \rho^{\alpha\varepsilon}\frac{\partial^2 u^{\alpha\varepsilon}_i}{\partial t^2} = f_i \quad \text{on } \Omega^\varepsilon_\alpha, \tag{2.1}$$

$$-\frac{\partial}{\partial x_i}\left(k^{\alpha\varepsilon}_{ij}\frac{\partial \theta^{\alpha\varepsilon}}{\partial x_j}\right) + T_0 b^{\alpha\varepsilon}_{ij}\frac{\partial e_{ij}(u^{\alpha\varepsilon})}{\partial t} + c^{\alpha\varepsilon}\frac{\partial \theta^{\alpha\varepsilon}}{\partial t} = r \quad \text{on } \Omega^\varepsilon_\alpha, \tag{2.2}$$

$$\sigma^{1\varepsilon}_{ij} n_j = \sigma^{2\varepsilon}_{ij} n_j = \varepsilon h^u_\varepsilon (u^{2\varepsilon}_i - u^{1\varepsilon}_i) \quad \text{on } \Gamma_\varepsilon, \tag{2.3}$$

$$k^{1\varepsilon}_{ij}\frac{\partial \theta^{1\varepsilon}}{\partial x_j} n_i = k^{2\varepsilon}_{ij}\frac{\partial \theta^{2\varepsilon}}{\partial x_j} n_i = \varepsilon h^\theta_\varepsilon (\theta^{2\varepsilon} - \theta^{1\varepsilon}) \quad \text{on } \Gamma_\varepsilon, \tag{2.4}$$

where f_i are the components of the vector field $f \in L^2(\Omega)^N$ which represent the forces, and $r \in L^2(\Omega)$ is the energy source. Moreover, we impose conditions on the boundary $\partial\Omega$,

$$u^{1\varepsilon} = 0, \quad \theta^{1\varepsilon} = 0 \quad \text{on } \partial\Omega, \tag{2.5}$$

and zero initial conditions, namely

$$u^{\alpha\varepsilon}(0, x) = 0, \quad \dot{u}^{\alpha\varepsilon}(0, x) = 0, \quad \theta^{\alpha\varepsilon}(0, x) = 0. \tag{2.6}$$

2.1 Variational formulation and a priori estimates

Let T be a real and strictly positive number. In the following we shall use the notations $\Omega_T = [0, T] \times \Omega$, $\Omega^\varepsilon_{T\alpha} = [0, T] \times \Omega^\varepsilon_\alpha$ and $\Gamma^T_\varepsilon = [0, T] \times \Gamma_\varepsilon$ and we introduce the spaces

$$V_{1\varepsilon} = \left\{v \in C^\infty(0, T; H^1(\Omega^\varepsilon_1))),\ v = 0 \text{ on } \partial\Omega \text{ and } v = 0 \text{ on } \{0\} \times \Omega\right\},$$

$$V_{2\varepsilon} = \left\{v \in C^\infty(0, T; H^1(\Omega^\varepsilon_2))),\ v = 0 \text{ on } \{0\} \times \Omega\right\},$$

$$W_\varepsilon = \left(V^N_{1\varepsilon} \times V^N_{2\varepsilon}\right) \times (V_{1\varepsilon} \times V_{2\varepsilon}). \tag{2.7}$$

An element of W_ε will be denoted $V = (v, w)$ where $v = (v^1, v^2) \in V^N_{1\varepsilon} \times V^N_{2\varepsilon}$ and $w = (w^1, w^2) \in V_{1\varepsilon} \times V_{2\varepsilon}$. In this space we introduce the weak formulation of problem (2.1)-(2.6) namely:

Find $U^\varepsilon = (u^\varepsilon, \theta^\varepsilon) \in W_\varepsilon$ such that

$$\mathcal{L}_\varepsilon(U^\varepsilon, V) = \mathcal{D}_\varepsilon\left((f, r), V\right),\ \forall V = (v, w) \in W_\varepsilon, \tag{2.8}$$

where, for each ε, $\mathcal{L}_\varepsilon : W_\varepsilon \times W_\varepsilon \to \mathbb{R}$ is a bilinear form defined by

$$\begin{aligned} \mathcal{L}_\varepsilon(U, V) = & \sum_{\alpha=1,2}\Bigg[\int_0^T\!\!\int_{\Omega^\varepsilon_\alpha}(t - T)\Big(\left(-a^{\alpha\varepsilon}_{ijkh}e_{kh}(u^\alpha) + b^{\alpha\varepsilon}_{ij}\theta^\alpha\right)e_{ij}(\dot{v}^\alpha) + \rho^{\alpha\varepsilon}\dot{u}^\alpha_i\ddot{v}^\alpha_i + +b^{\alpha\varepsilon}_{ij}e_{ij}(u^\alpha)\dot{w}^\alpha + \\ & + \frac{1}{T_0}c^{\alpha\varepsilon}\theta^\alpha\dot{w}^\alpha\Big) + \rho^{\alpha\varepsilon}\dot{u}^\alpha_i\dot{v}^\alpha_i + b^{\alpha\varepsilon}_{ij}e_{ij}(u^\alpha)w^\alpha + \frac{1}{T_0}c^{\alpha\varepsilon}\theta^\alpha w^\alpha + \frac{1}{T_0}\int_0^t k^{\alpha\varepsilon}_{ij}\frac{\partial\theta^\alpha}{\partial x_j}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{d}s\Bigg] - \\ & - \varepsilon\int_0^T\!\!\int_{\Gamma_\varepsilon}(t - T)h^u_\varepsilon(u^2_i - u^1_i)(\dot{v}^2_i - \dot{v}^1_i) - \frac{\varepsilon}{T_0}\int_0^T\!\!\int_{\Gamma_\varepsilon}(t - T)h^\theta_\varepsilon(\theta^2 - \theta^1)(w^2 - w^1), \end{aligned} \tag{2.9}$$

with $U = (u, \theta)$, $V = (v, w)$, and $\mathcal{D}_\varepsilon : \left(L^2(\Omega)^N \times L^2(\Omega)\right) \times W_\varepsilon \to \mathbb{R}$ defined by

$$\mathcal{D}_\varepsilon\left((f, r), V\right) = -\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega^\varepsilon_\alpha}(t - T)\Big(f_i\dot{v}^\alpha_i + \frac{1}{T_0}rw^\alpha\Big). \tag{2.10}$$

We introduce now the Hilbert space $\mathcal{W}_\varepsilon$ which is the completion of W_ε in norm $\|\cdot\|$ generated by the scalar product

$$\begin{aligned}(U,V)_{W_\varepsilon} = \quad & \sum_{\alpha=1,2}\left[\int_0^T\!\!\int_{\Omega_\alpha^\varepsilon} u_i^\alpha v_i^\alpha + \dot{u}_i^\alpha \dot{v}_i^\alpha + e_{ij}(u^\alpha)e_{ij}(v^\alpha) + \theta^\alpha w^\alpha + \int_0^t \frac{\partial \theta^\alpha}{\partial x_i}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{d}s\right] \\ & + \varepsilon\int_0^T\!\!\int_{\Gamma_\varepsilon}(u_i^2-u_i^1)(v_i^2-v_i^1) + \varepsilon\int_0^T\!\!\int_{\Gamma_\varepsilon}\int_0^t(\theta^2-\theta^1)(w^2-w^1)\,\mathrm{d}s.\end{aligned} \tag{2.11}$$

and one can see that $\mathcal{L}_\varepsilon$ can be extended by continuity to the entire space $\mathcal{W}_\varepsilon \times \mathcal{W}_\varepsilon$, and $\mathcal{D}_\varepsilon$ can be extended to $\big(L^2(\Omega)^N \times L^2(\Omega)\big) \times \mathcal{W}_\varepsilon$.

Theorem 2.1. *The problem (2.8) has a unique soluiton. Moreover, there exists a constant $C>0$, independent of ε, such that:*

$$\|u_i^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \|\dot{u}_i^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \|\nabla u_i^{\alpha\varepsilon}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \|\theta^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \tag{2.12}$$

$$\left\|\int_0^t(\nabla\theta^{\varepsilon\alpha})^2\right\|_{L^1(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \|u_i^{2\varepsilon}-u_i^{1\varepsilon}\|_{L^2(\Gamma_\varepsilon^T)} \leqslant C\varepsilon^{-1/2}, \quad \left\|\int_0^t\big(\theta^{2\varepsilon}-\theta^{1\varepsilon}\big)^2\right\|_{L^1(\Gamma_\varepsilon^T)} \leqslant C\varepsilon^{-1/2}. \tag{2.13}$$

2.2 The homogenization process

In this section we shall use the notation

$$\begin{aligned} W = \quad & H^2(0,T;H_0^1(\Omega))^N \times L^\infty(0,T;L^2(\Omega;H^1_{per}(Y_1))^N \times H^2(0,T;L^2(\Omega))^N \times \\ & \times H^1(0,T;H_0^1(\Omega)) \times L^\infty(0,T;L^2(\Omega;H^1_{per}(Y_1)) \times H^1(0,T;L^2(\Omega)).\end{aligned}$$

Theorem 2.2. *If $(u^\varepsilon,\theta^\varepsilon)\in\mathcal{W}_\varepsilon$ is the solution of ptoblem (2.1)-(2.6), where $u^\varepsilon=(u^{1\varepsilon},u^{2\varepsilon})$ and $\theta^\varepsilon=(\theta^{1\varepsilon},\theta^{2\varepsilon})$, then*

$$\begin{aligned}
&\tilde{u}^{\alpha\varepsilon} \overset{*}{\rightharpoonup} |Y_\alpha|\cdot u^\alpha \quad \textit{weakly* in} \quad L^\infty(0,T;L^2(\Omega))^N, \\
&\tilde{\theta}^{\alpha\varepsilon} \overset{*}{\rightharpoonup} |Y_\alpha|\cdot \theta^\alpha \quad \textit{weakly* in} \quad L^\infty(0,T;L^2(\Omega)), \\
&\mathcal{T}_\alpha^\varepsilon(u^{\alpha\varepsilon}) \overset{*}{\rightharpoonup} u^\alpha \quad \textit{weakly* in} \quad L^\infty(0,T;L^2(\Omega;H^1(Y_\alpha)))^N, \\
&\mathcal{T}_1^\varepsilon(e_{kh}(u^{1\varepsilon})) \overset{*}{\rightharpoonup} e_{kh}(u^1)+e^y_{kh}(\widehat{u}^1) \quad \textit{weakly* in} \quad L^\infty(0,T;L^2(\Omega\times Y_1)), \\
&\mathcal{T}_2^\varepsilon(e_{kh}(u^{2\varepsilon})) \overset{*}{\rightharpoonup} 0 \quad \textit{weakly* in} \quad L^\infty(0,T;L^2(\Omega\times Y_2)), \\
&\mathcal{T}_\alpha^\varepsilon(\theta^{\alpha\varepsilon}) \overset{*}{\rightharpoonup} \theta^\alpha \quad \textit{weakly* in} \quad L^\infty(0,T;L^2(\Omega;H^1(Y_\alpha))), \\
&\mathcal{T}_1^\varepsilon(\nabla\theta^{1\varepsilon}) \overset{*}{\rightharpoonup} \nabla\theta^1+\nabla_y\widehat{\theta}^1 \quad \textit{weakly* in} \quad L^\infty(0,T;L^2(\Omega\times Y_1)), \\
&\mathcal{T}_2^\varepsilon(\nabla\theta^{2\varepsilon}) \overset{*}{\rightharpoonup} 0 \quad \textit{weakly* in} \quad L^\infty(0,T;L^2(\Omega\times Y_2)),
\end{aligned} \tag{2.14}$$

where $(u^1,\widehat{u}^1,u^2,\theta^1,\widehat{\theta}^1,\theta^2)\in W$, is the unique solution of problem

$$\begin{aligned}
&\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\left[a^1_{ijkh}\Big(e_{kh}(u^1)+e^y_{kh}(\widehat{u}^1)\Big)-b^1_{ij}\theta^1\right]\Big(\dot{e}_{ij}(\varphi^1)+\dot{e}^y_{ij}(\Phi^1)\Big)+ \\
&\quad + \sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\Big[\rho^\alpha\ddot{u}_i^\alpha\dot{\varphi}_i^\alpha+\frac{1}{T_0}c^\alpha\dot{\theta}^\alpha q^\alpha\Big]+ \\
&+\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)k^1_{ij}\Big(\frac{\partial\theta^1}{\partial x_j}+\frac{\partial\widehat{\theta}^1}{\partial y_j}\Big)\Big(\frac{\partial q^1}{\partial x_i}+\frac{\partial Q^1}{\partial y_i}\Big)+\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)b^1_{ij}\Big(\dot{e}_{ij}(u^1)+\dot{e}^y_{ij}(\widehat{u}^1)\Big)q^1+ \\
&\quad +\int_0^T\!\!\int_{\Omega\times\Gamma}(t-T)\Big[h^u(u_i^2-u_i^1)(\dot{\varphi}_i^2-\dot{\varphi}_i^1)+\frac{1}{T_0}h^\theta(\theta_i^2-\theta_i^1)(q_i^2-q_i^1)\Big]= \\
&= \sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\Big(f_i\dot{\varphi}_i^\alpha+\frac{1}{T_0}rq^\alpha\Big), \qquad \forall(\varphi^1,\Phi^1,\varphi^2,q^1,Q^1,q^2)\in W.
\end{aligned} \tag{2.15}$$

Moreover, for $\alpha\in\{1,2\}$ and for almost $x\in\Omega$ we have $u^\alpha(0,x)=0$, $\quad \dot{u}^\alpha(0,x)=0$, $\quad \theta^\alpha(0,x)=0$.

We introduce now the unique solutions $z^1, w_1^{lm}, \chi^1 \in \tilde{H}^1_{per}(Y_1)^N$ $(l,m=1,\dots,N)$, of the cell problems

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\Big(a^1_{ijkh}\dfrac{\partial z^1_k}{\partial y_h} - b^1_{ij}\Big) = 0 \quad \text{in } Y_1 \\ \Big(a^1_{ijkh}\dfrac{\partial z^1_k}{\partial y_h} - b^1_{ij}\Big)n_j = 0 \quad \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\dfrac{\partial}{\partial y_j}\Big(a^1_{ijlm} + a^1_{ijkh}\dfrac{\partial w^{lm}_{1k}}{\partial y_h}\Big) = 0 \quad \text{in } Y_1 \\ \Big(a^1_{ijlm} + a^1_{ijkh}\dfrac{\partial w^{lm}_{1k}}{\partial y_h}\Big)n_j = 0 \quad \text{on } \Gamma, \end{cases} \tag{2.16}$$

respectively

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\Big(k^1_{ik} + k^1_{ij}\dfrac{\partial \chi^1_k}{\partial y_j}\Big) = 0 \quad \text{in } Y_1 \\ \Big(k^1_{ik} + k^1_{ij}\dfrac{\partial \chi^1_k}{\partial y_j}\Big)n_i = 0 \quad \text{on } \Gamma, \end{cases} \tag{2.17}$$

and one can see that

$$\widehat{u}^1_k(t,x,y) = \frac{\partial u^1_l}{\partial x_m}(t,x)\cdot w^{lm}_{1k}(y) + \theta^1(t,x)\cdot z^1_k(y). \tag{2.18}$$

$$\widehat{\theta}^1(t,x,y) = \frac{\partial \theta^1}{\partial x_k}(t,x)\cdot \chi^1_k(y). \tag{2.19}$$

we define now the homogenized coefficients

$$a^{1*}_{ijlm} = \int_{Y_1} a^1_{ijlm} + a^1_{ijkh}\frac{\partial w^{lm}_{1k}}{\partial y_h}, \quad b^{1*}_{lm} = \int_{Y_1} b^1_{lm} + b^1_{ij}\frac{\partial w^{lm}_{1i}}{\partial y_j}, \quad k^{1*}_{ik} = \int_{Y_1} k^1_{ik} + k^1_{ij}\frac{\partial \chi^1_k}{\partial y_j}, \quad \gamma^{1*} = \int_{Y_1} b^1_{ij}\frac{\partial z^1_i}{\partial y_j}. \tag{2.20}$$

and we prove the next theorem:

Theorem 2.3. *If $(u^\varepsilon, \theta^\varepsilon) \in \mathcal{W}_\varepsilon$ is the solution of problem (2.1)-(2.6), where $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ and $\theta^\varepsilon = (\theta^{1\varepsilon}, \theta^{2\varepsilon})$, then for $\alpha \in \{1,2\}$*

$$\tilde{u}^{\alpha\varepsilon} \stackrel{*}{\rightharpoonup} |Y_\alpha|\cdot u^\alpha \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega))^N, \tag{2.21}$$

$$\tilde{\theta}^{\alpha\varepsilon} \stackrel{*}{\rightharpoonup} |Y_\alpha|\cdot \theta^\alpha \quad \textit{weakly* in } L^\infty(0,T;L^2(\Omega)), \tag{2.22}$$

where (u,θ) with $u=(u^1,u^2)$ and $\theta=(\theta^1,\theta^2)$ is the unique solution of problem

$$-\frac{\partial}{\partial x_j}\Big(a^{1*}_{ijkh}\frac{\partial u^1_k}{\partial x_h} - b^{1*}_{ij}\theta^1\Big) + \langle\rho^1\rangle_{Y_1}\frac{\partial^2 u^1_i}{\partial t^2} - H^u(u^2_i - u^1_i) = |Y_1|\, f_i \quad \textit{in } \Omega, \tag{2.23}$$

$$\langle\rho^2\rangle_{Y_2}\frac{\partial^2 u^2_i}{\partial t^2} + H^u(u^2_i - u^1_i) = |Y_2|\, f_i \quad \textit{in } \Omega, \tag{2.24}$$

$$-\frac{\partial}{\partial x_i}\Big(k^{1*}_{ij}\frac{\partial \theta^1}{\partial x_j}\Big) + T_0 b^{1*}_{ij}\frac{\partial e_{ij}(u^1)}{\partial t} + \Big(T_0\gamma^{1*} + \langle c^1\rangle_{Y_1}\Big)\frac{\partial \theta^1}{\partial t} - H^\theta(\theta^2 - \theta^1) = |Y_1|\, r \quad \textit{in } \Omega, \tag{2.25}$$

$$\langle c^2\rangle_{Y_2}\frac{\partial \theta^2}{\partial t} + H^\theta(\theta^2_i - \theta^1_i) = |Y_2|\, r \quad \textit{in } \Omega, \tag{2.26}$$

$$u^1 = 0, \quad \theta^1 = 0 \quad \textit{on } \partial\Omega, \tag{2.27}$$

with the initial conditions

$$u^\alpha(0,x) = 0, \quad \dot{u}^\alpha(0,x) = 0, \quad \theta^\alpha(0,x) = 0. \tag{2.28}$$

where $H^u = \int_\Gamma h^u$ and $H^\theta = \int_\Gamma h^\theta$.

3 The thermoelastic model in a double porosity medium with jumps in displacements and temperatures

In this chapter we study again problem (2.1)-(2.5) considered in Chapter 2 but this time the domain Ω is occupied by a double porosity medium similar to the one considered in Chapter 1. As we expected, the results obtained here are a combination between the results obtained in homogenization of the elastic problem considered in a double porosity medium, from Chapter 1, and the results in the homogenization of a diffusion problem stated in a domain occupied by a clasic medium. As in Chapter 1, it is interesting to see that although the tensors A^2

and B^2 do not appear in the homogenized problem, they appear in the limit of $\tilde{u}^{2\varepsilon}$ through two terms ξ^l and ζ, respectively which are part of the mentioned limit.

In what follows we shal use the same notations as in Chapter 2 and the coefficients of the problem will have the same properties. Unlike the precedent chapter we will consider that $A^{2\varepsilon}(x) = \varepsilon^2 A^2(x/\varepsilon)$, and $\rho^{2\varepsilon}(x) = \varepsilon\rho^2(x/\varepsilon)$. The space $\mathcal{W}_\varepsilon$ represents this time the completion of W_ε in norm $\|\cdot\|$ generated by the scalar product

$$\begin{aligned}(U,V)_{W_\varepsilon} = \; & \int_0^T\!\!\int_{\Omega_1^\varepsilon} e_{ij}(u^1)e_{ij}(v^1) + \varepsilon^2 \int_0^T\!\!\int_{\Omega_2^\varepsilon} e_{ij}(u^2)e_{ij}(v^2) + \sum_{\alpha=1,2}\left[\int_0^T\!\!\int_{\Omega_\alpha^\varepsilon} \dot{u}_i^\alpha \dot{v}_i^\alpha + \theta^\alpha w^\alpha + \int_0^t \frac{\partial \theta^\alpha}{\partial x_i}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{d}s\right] + \\ & + \varepsilon \int_0^T\!\!\int_{\Gamma_\varepsilon} (u_i^2 - u_i^1)(v_i^2 - v_i^1) + \varepsilon \int_0^T\!\!\int_{\Gamma_\varepsilon}\int_0^t (\theta^2-\theta^1)(w^2-w^1)\,\mathrm{d}s.\end{aligned} \tag{3.1}$$

and the forms $\mathcal{L}_\varepsilon$ and $\mathcal{D}_\varepsilon$ can still be extended by continuity to spaces încă se pot prelungi prin continuitate la $\mathcal{W}_\varepsilon \times \mathcal{W}_\varepsilon$ and $\Big(L^2(\Omega)^N \times L^2(\Omega)\Big) \times \mathcal{W}_\varepsilon$, respectively.

Theorem 3.1. *Există o constantă $C > 0$, independentă de ε pentru care:*

$$\|u_i^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \|\dot{u}_i^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \|\nabla u_i^{1\varepsilon}\|_{L^2(\Omega_{T1}^\varepsilon)} \leqslant C, \quad \varepsilon\|\nabla u_i^{2\varepsilon}\|_{L^2(\Omega_{T2}^\varepsilon)} \leqslant C, \tag{3.2}$$

$$\|\theta^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \left\|\int_0^t (\nabla\theta^{\varepsilon\alpha})^2\right\|_{L^1(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \tag{3.3}$$

$$\|u_i^{2\varepsilon} - u_i^{1\varepsilon}\|_{L^2(\Gamma_\varepsilon^T)} \leqslant C\varepsilon^{-1/2}, \quad \left\|\int_0^t (\theta^{2\varepsilon} - \theta^{1\varepsilon})^2\right\|_{L^1(\Gamma_\varepsilon^T)} \leqslant C\varepsilon^{-1/2}. \tag{3.4}$$

3.1 Homogenization results

We consider

$$\begin{aligned} W = \; & H^2(0,T;H_0^1(\Omega))^N \times L^\infty(0,T;L^2(\Omega;H_{per}^1(Y_1))^N \times L^\infty(0,T;L^2(\Omega;H^1(Y_2)))^N \times \\ & \times H^1(0,T;H_0^1(\Omega)) \times L^\infty(0,T;L^2(\Omega;H_{per}^1(Y_1)) \times H^1(0,T;L^2(\Omega)),\end{aligned} \tag{3.5}$$

Theorem 3.2. *If $(u^\varepsilon, \theta^\varepsilon) \in \mathcal{W}_\varepsilon$ is the solution of problem (2.1)-(2.6), with $A^{1\varepsilon}(x) = A^1(x/\varepsilon)$, $A^{2\varepsilon}(x) = \varepsilon^2 A^2(x/\varepsilon)$, $\rho^{1\varepsilon}(x) = \rho^1(x/\varepsilon)$ and $\rho^{2\varepsilon}(x) = \varepsilon\rho^2(x/\varepsilon)$ where $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ and $\theta^\varepsilon = (\theta^{1\varepsilon}, \theta^{2\varepsilon})$, then*

$$\begin{aligned}
&\tilde{u}^{1\varepsilon} \stackrel{*}{\rightharpoonup} |Y_1|\cdot u^1 \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega))^N,\\
&\tilde{u}^{2\varepsilon} \stackrel{*}{\rightharpoonup} |Y_2|\cdot \langle \widehat{u}^2\rangle_{Y_2} \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega))^N,\\
&\tilde{\theta}^{\alpha\varepsilon} \stackrel{*}{\rightharpoonup} |Y_\alpha|\cdot \theta^\alpha \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega)),\\
&\mathcal{T}_\alpha^\varepsilon(u^{1\varepsilon}) \stackrel{*}{\rightharpoonup} u^1 \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega;H^1(Y_1)))^N,\\
&\mathcal{T}_\alpha^\varepsilon(u^{2\varepsilon}) \stackrel{*}{\rightharpoonup} \widehat{u}^2 \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega;H^1(Y_2)))^N,\\
&\mathcal{T}_1^\varepsilon(e_{kh}(u^{1\varepsilon})) \stackrel{*}{\rightharpoonup} e_{kh}(u^1) + e_{kh}^y(\widehat{u}^1) \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_1)),\\
&\varepsilon\mathcal{T}_2^\varepsilon(e_{kh}(u^{2\varepsilon})) \stackrel{*}{\rightharpoonup} e_{kh}^y(\widehat{u}^2) \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_2)),\\
&\mathcal{T}_\alpha^\varepsilon(\theta^{\alpha\varepsilon}) \stackrel{*}{\rightharpoonup} \theta^\alpha \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega;H^1(Y_\alpha))),\\
&\mathcal{T}_1^\varepsilon(\nabla\theta^{1\varepsilon}) \stackrel{*}{\rightharpoonup} \nabla\theta^1 + \nabla_y\widehat{\theta}^1 \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_1)),\\
&\mathcal{T}_2^\varepsilon(\nabla\theta^{2\varepsilon}) \stackrel{*}{\rightharpoonup} 0 \quad \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_2)),
\end{aligned} \tag{3.6}$$

where $(u^1, \widehat{u}^1, \widehat{u}^2, \theta^1, \widehat{\theta}^1, \theta^2) \in W$, is the unique solution of problem

$$\int_0^T\!\!\int_{\Omega\times Y_1} (t-T)\left[a_{ijkh}^1\Big(e_{kh}(u^1) + e_{kh}^y(\widehat{u}^1)\Big) - b_{ij}^1\theta^1\right]\Big(\dot{e}_{ij}(\varphi^1) + \dot{e}_{ij}^y(\Phi^1)\Big) +$$

$$+\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)\left[a^2_{ijkh}e^y_{kh}(\widehat{u}^2)-b^2_{ij}\theta^2\right]\dot{e}^y_{ij}(\Phi^2)+\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\rho^1\ddot{u}^1_i\dot{\varphi}^1_i+$$
$$+\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)b^1_{ij}\Big(\dot{e}_{ij}(u^1)+\dot{e}^y_{ij}(\widehat{u}^1)\Big)q^1+\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)b^2_{ij}\dot{e}^y_{ij}(\widehat{u}^2)q^2+ \tag{3.7}$$
$$+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)c^\alpha\dot{\theta}^\alpha q^\alpha+\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)k^1_{ij}\Big(\frac{\partial\theta^1}{\partial x_j}+\frac{\partial\widehat{\theta}^1}{\partial y_j}\Big)\Big(\frac{\partial q^1}{\partial x_i}+\frac{\partial Q^1}{\partial y_i}\Big)+$$
$$+\int_0^T\!\!\int_{\Omega\times\Gamma}(t-T)\Big[h^u(\widehat{u}^2_i-u^1_i)(\dot{\Phi}^2_i-\dot{\varphi}^1_i)+\frac{1}{T_0}h^\theta(\theta^2_i-\theta^1_i)(q^2_i-q^1_i)\Big]=$$
$$=\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\Big(f_i\dot{\varphi}^1_i+\frac{1}{T_0}rq^1\Big)+\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)\Big(f_i\dot{\Phi}^2_i+\frac{1}{T_0}rq^2\Big),$$
$$\forall(\varphi^1,\Phi^1,\Phi^2,q^1,Q^1,q^2)\in W.$$

Moreover, for almost every $x\in\Omega$ we have $u^1(0,x)=0,\quad \dot{u}^1(0,x)=0,\quad \theta^\alpha(0,x)=0.$

We find the expresions of $\widehat{u}^1$, $\widehat{u}^2$, $\widehat{\theta}^1$ thus (2.18)-(2.19) still hold, moreover

$$\widehat{u}^2_k(t,x,y)=u^1_k(t,x)+f_l(x)w^l_{2k}(y)+\theta^2(t,x)z^2_k(y). \tag{3.8}$$

where $z^2, w^l_2\in H^1(Y_2)^N$ are the unique solutions of cell problems

$$\begin{cases}-\dfrac{\partial}{\partial y_j}\Big(a^2_{ijkh}\dfrac{\partial w^l_{2k}}{\partial y_h}\Big)=\delta_{il} & \text{in } Y_2\\[2mm] a^2_{ijkh}\dfrac{\partial w^l_{2k}}{\partial y_h}n_j=h^u w^l_{2i} & \text{on } \Gamma,\end{cases}\qquad \begin{cases}-\dfrac{\partial}{\partial y_j}\Big(a^2_{ijkh}\dfrac{\partial z^2_k}{\partial y_h}-b^2_{ij}\Big)=0 & \text{in } Y_2\\[2mm] \Big(a^2_{ijkh}\dfrac{\partial z^2_k}{\partial y_h}-b^2_{ij}\Big)n_j=0 & \text{on } \Gamma.\end{cases} \tag{3.9}$$

We define

$$\gamma^{2*}=\int_{Y_2}b^2_{ij}\frac{\partial z^2_i}{\partial y_j} \tag{3.10}$$

an introducing (2.18), (2.19), (2.20), (3.8) and (3.10) into the limit problem (3.7), we get the homogenized problem in Ω. More exactly, we prove the following theorem:

Theorem 3.3. *If $(u^\varepsilon,\theta^\varepsilon)\in\mathcal{W}_\varepsilon$ is the solution of problem (2.1)-(2.6) with $A^{1\varepsilon}(x)=A^1(x/\varepsilon)$, $A^{2\varepsilon}(x)=\varepsilon^2A^2(x/\varepsilon)$, $\rho^{1\varepsilon}(x)=\rho^1(x/\varepsilon)$ and $\rho^{2\varepsilon}(x)=\varepsilon\rho^2(x/\varepsilon)$, where $u^\varepsilon=(u^{1\varepsilon},u^{2\varepsilon})$ and $\theta^\varepsilon=(\theta^{1\varepsilon},\theta^{2\varepsilon})$, then we have*

$$\tilde{u}^{1\varepsilon}\overset{*}{\rightharpoonup}|Y_1|\cdot u^1\quad \textit{weakly* in}\quad L^\infty(0,T;L^2(\Omega))^N, \tag{3.11}$$
$$\tilde{u}^{2\varepsilon}\overset{*}{\rightharpoonup}|Y_2|\cdot u^1+f_l\cdot\xi^l+\theta^2\cdot\zeta\ \textit{weakly* in}\quad L^\infty(0,T;L^2(\Omega))^N, \tag{3.12}$$
$$\tilde{\theta}^{\alpha\varepsilon}\overset{*}{\rightharpoonup}|Y_\alpha|\cdot\theta^\alpha\quad \textit{weakly* in}\quad L^\infty(0,T;L^2(\Omega)),\quad \forall\alpha\in\{1,2\}, \tag{3.13}$$

where (u,θ) with $u=(u^1,u^2)$ and $\theta=(\theta^1,\theta^2)$ is the unique solution of problem

$$-\frac{\partial}{\partial x_j}\Big(a^{1*}_{ijkh}\frac{\partial u^1_k}{\partial x_h}-b^{1*}_{ij}\theta^1\Big)+\langle\rho^1\rangle_{Y_1}\frac{\partial^2u^1_i}{\partial t^2}=f_i\quad \textit{in } \Omega, \tag{3.14}$$
$$-\frac{\partial}{\partial x_i}\Big(k^{1*}_{ij}\frac{\partial\theta^1}{\partial x_j}\Big)+T_0b^{1*}_{ij}\frac{\partial e_{ij}(u^1)}{\partial t}+\Big(T_0\gamma^{1*}+\langle c^1\rangle_{Y_1}\Big)\frac{\partial\theta^1}{\partial t}-H^\theta(\theta^2_i-\theta^1_i)=|Y_1|\,r\quad \textit{in } \Omega, \tag{3.15}$$
$$\Big(T_0\gamma^{2*}+\langle c^2\rangle_{Y_2}\Big)\frac{\partial\theta^2}{\partial t}+H^\theta(\theta^2_i-\theta^1_i)=|Y_2|\,r\quad \textit{in } \Omega, \tag{3.16}$$
$$u^1=0,\quad \theta^1=0\quad \textit{on } \partial\Omega, \tag{3.17}$$

with the initial conditions

$$u^\alpha(0,x)=0,\quad \dot{u}^\alpha(0,x)=0,\quad \theta^\alpha(0,x)=0, \tag{3.18}$$

where $H^\theta=\int_\Gamma h^\theta$ and the components of vector fields ξ^l and ζ being $\xi^l_i=\int_{Y_2}w^l_{2i}$ and $\zeta_i=\int_{Y_2}z^2_i$, respectively.

4 The thermoelastic model in a double porosity medium with jumps in displacements and continuity in temperatures

In this chapter we change the jump of temperatures condition on the interface Γ_ε, from the problem studied in Chapter 3. More exactly, on each of the two components Ω_1^ε and Ω_2^ε of the domain Ω occupied by a double porosity medium, we consider the ecuations

$$-\frac{\partial \sigma_{ij}^{\alpha\varepsilon}}{\partial x_j} + \rho^{\alpha\varepsilon}\frac{\partial^2 u_i^{\alpha\varepsilon}}{\partial t^2} = f_i \tag{4.1}$$

$$-\frac{\partial}{\partial x_i}\left(k_{ij}^{\alpha\varepsilon}\frac{\partial \theta^{\alpha\varepsilon}}{\partial x_j}\right) + T_0 b_{ij}^{\alpha\varepsilon}\frac{\partial e_{ij}(u^{\alpha\varepsilon})}{\partial t} + c^{\alpha\varepsilon}\frac{\partial \theta^{\alpha\varepsilon}}{\partial t} = r \tag{4.2}$$

along the conditions on the interface Γ_ε and the boundary $\partial\Omega$, respectively

$$\sigma_{ij}^{1\varepsilon} n_j = \sigma_{ij}^{2\varepsilon} n_j = \varepsilon h_\varepsilon^u (u_i^{2\varepsilon} - u_i^{1\varepsilon}) \quad \text{on } \Gamma_\varepsilon, \tag{4.3}$$

$$k_{ij}^{1\varepsilon}\frac{\partial \theta^{1\varepsilon}}{\partial x_j} n_i = k_{ij}^{2\varepsilon}\frac{\partial \theta^{2\varepsilon}}{\partial x_j} n_i \quad \text{on } \Gamma_\varepsilon, \tag{4.4}$$

$$\theta^{1\varepsilon} = \theta^{2\varepsilon} \quad \text{on } \Gamma_\varepsilon, \tag{4.5}$$

$$u^{1\varepsilon} = 0, \quad \theta^{1\varepsilon} = 0 \quad \text{on } \partial\Omega, \tag{4.6}$$

and the initial conditions

$$u^{\alpha\varepsilon}(0,x) = 0, \quad \dot u^{\alpha\varepsilon}(0,x) = 0, \quad \theta^{\alpha\varepsilon}(0,x) = 0. \tag{4.7}$$

As in Chapter 3 we consider that

$$A^{1\varepsilon}(x) = A^1(x/\varepsilon), \qquad A^{2\varepsilon}(x) = \varepsilon^2 A^2(x/\varepsilon), \qquad B^{1\varepsilon}(x) = B^1(x/\varepsilon, \qquad B^{2\varepsilon}(x) = \varepsilon B^2(x/\varepsilon),$$
$$\rho^{1\varepsilon}(x) = \rho^1(x/\varepsilon), \qquad \rho^{2\varepsilon}(x) = \varepsilon\rho^2(x/\varepsilon), \qquad K^{\alpha\varepsilon}(x) = K^\alpha(x/\varepsilon), \qquad c^{\alpha\varepsilon}(x) = c^\alpha(x/\varepsilon).$$

The functional space used in this chapter will also be the space $W_\varepsilon$defined in Chapter 2 by (2.7), and the variational formulation of problem (4.1)-(4.7) is:

Find $U^\varepsilon = (u^\varepsilon, \theta^\varepsilon) \in W_\varepsilon$ such that

$$\mathcal{L}_\varepsilon(U^\varepsilon, V) = \mathcal{D}_\varepsilon\left((f,r),V\right), \ \forall V = (v,w) \in W_\varepsilon, \tag{4.8}$$

where for each ε, the bilinear form $\mathcal{L}_\varepsilon : W_\varepsilon \times W_\varepsilon \to \mathbb{R}$ is defined this time by:

$$\begin{aligned} \mathcal{L}_\varepsilon(U,V) = \ & \sum_{\alpha=1,2}\Bigg[\int_0^T\!\!\int_{\Omega_\alpha^\varepsilon}(t-T)\Big(\left(-a_{ijkh}^{\alpha\varepsilon}e_{kh}(u^\alpha) + b_{ij}^{\alpha\varepsilon}\theta^\alpha\right)e_{ij}(\dot v^\alpha) + \rho^{\alpha\varepsilon}\dot u_i^\alpha \ddot v_i^\alpha + + b_{ij}^{\alpha\varepsilon}e_{ij}(u^\alpha)\dot w^\alpha + \\ & + \frac{1}{T_0}c^{\alpha\varepsilon}\theta^\alpha \dot w^\alpha\Big) + \rho^{\alpha\varepsilon}\dot u_i^\alpha \dot v_i^\alpha + b_{ij}^{\alpha\varepsilon}e_{ij}(u^\alpha)w^\alpha + \frac{1}{T_0}c^{\alpha\varepsilon}\theta^\alpha w^\alpha + \frac{1}{T_0}\int_0^t k_{ij}^{\alpha\varepsilon}\frac{\partial\theta^\alpha}{\partial x_j}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{ds}\Bigg] - \\ & -\varepsilon\int_0^T\!\!\int_{\Gamma_\varepsilon}(t-T)h_\varepsilon^u(u_i^2 - u_i^1)(\dot v_i^2 - \dot v_i^1), \end{aligned} \tag{4.9}$$

and $\mathcal{D}_\varepsilon : \left(L^2(\Omega)^N \times L^2(\Omega)\right) \times W_\varepsilon \to \mathbb{R}$ is given by (2.10). The space $\mathcal{W}_\varepsilon$ is obtained this time by completion of W_ε in norm generated by the scalar product

$$\begin{aligned} (U,V)_{W_\varepsilon} = \ & \int_0^T\!\!\int_{\Omega_1^\varepsilon} e_{ij}(u^1)e_{ij}(v^1) + \varepsilon^2\int_0^T\!\!\int_{\Omega_2^\varepsilon} e_{ij}(u^2)e_{ij}(v^2) + \\ & + \sum_{\alpha=1,2}\left[\int_0^T\!\!\int_{\Omega_\alpha^\varepsilon}\dot u_i^\alpha \dot v_i^\alpha + \theta^\alpha w^\alpha + \int_0^t \frac{\partial\theta^\alpha}{\partial x_i}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{ds}\right] + \varepsilon\int_0^T\!\!\int_{\Gamma_\varepsilon}(u_i^2 - u_i^1)(v_i^2 - v_i^1). \end{aligned} \tag{4.10}$$

and again the forms $\mathcal{L}_\varepsilon(\cdot,\cdot)$ and $\mathcal{D}_\varepsilon(\cdot,\cdot)$ can be extended by continuity to $\mathcal{W}_\varepsilon \times \mathcal{W}_\varepsilon$ and $\left(L^2(\Omega)^N \times L^2(\Omega)\right) \times \mathcal{W}_\varepsilon$, respectively.

Theorem 4.1. *Problem (4.8) has a unique solution. Moreover, the exists a constant $C > 0$, independent of ε, such that:*

$$\|u_i^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \qquad \|\dot u_i^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \qquad \|\nabla u_i^{\alpha\varepsilon}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \tag{4.11}$$

$$\|\theta^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \qquad \left\|\int_0^t (\nabla\theta^{\varepsilon\alpha})^2\right\|_{L^1(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \qquad \|u_i^{2\varepsilon} - u_i^{1\varepsilon}\|_{L^2(\Gamma_\varepsilon^T)} \leqslant C\varepsilon^{-1/2}. \tag{4.12}$$

4.1 Homogenization results

Theorem 4.2. *If* $(u^\varepsilon, \theta^\varepsilon) \in \mathcal{W}_\varepsilon$ *is the unique solution of* (4.1)-(4.7)*, where* $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ *and* $\theta^\varepsilon = (\theta^{1\varepsilon}, \theta^{2\varepsilon})$*, then convergences* (3.6) *hold, where* $(u^1, \widehat{u}^1, \widehat{u}^2, \theta^1, \widehat{\theta}^1, \theta^2) \in W$ *defined by* (3.5)*, is the unique solution of problem*

$$\begin{aligned}
&\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\left[a^1_{ijkh}\Big(e_{kh}(u^1)+e^y_{kh}(\widehat{u}^1)\Big)-b^1_{ij}\theta^1\right]\Big(\dot{e}_{ij}(\varphi^1)+\dot{e}^y_{ij}(\Phi^1)\Big)+\\
&+\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)\left[a^2_{ijkh}e^y_{kh}(\widehat{u}^2)-b^2_{ij}\theta^2\right]\dot{e}^y_{ij}(\Phi^2)+\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\rho^1\ddot{u}^1_i\dot{\varphi}^1_i+\\
&+\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)b^1_{ij}\Big(\dot{e}_{ij}(u^1)+\dot{e}^y_{ij}(\widehat{u}^1)\Big)q^1+\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)b^2_{ij}\dot{e}^y_{ij}(\widehat{u}^2)q^2+\\
&+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)c^\alpha\dot{\theta}^\alpha q^\alpha+\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)k^1_{ij}\Big(\frac{\partial\theta^1}{\partial x_j}+\frac{\partial\widehat{\theta}^1}{\partial y_j}\Big)\Big(\frac{\partial q^1}{\partial x_i}+\frac{\partial Q^1}{\partial y_i}\Big)+\\
&+\int_0^T\!\!\int_{\Omega\times\Gamma}(t-T)h^u(\widehat{u}^2_i-u^1_i)(\dot{\Phi}^2_i-\dot{\varphi}^1_i)=\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\Big(f_i\dot{\varphi}^1_i+\frac{1}{T_0}rq^1\Big)+\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)\Big(f_i\dot{\Phi}^2_i+\frac{1}{T_0}rq^2\Big),\\
&\forall(\varphi^1,\Phi^1,\Phi^2,q^1,Q^1,q^2)\in W.
\end{aligned}\tag{4.13}$$

Moreover, for almost every $x \in \Omega$ *we have*

$$u^1(0,x)=0,\quad \dot{u}^1(0,x)=0,\quad \theta^\alpha(0,x)=0. \tag{4.14}$$

We prove that the expressions (2.18), (2.19), (3.8) of functions $\widehat{u}^1$, $\widehat{\theta}^1$ and $\widehat{u}^2$, respectively, still hold and introducing them into the limit problem (4.13) we get the homogenized problem in Ω.

Theorem 4.3. *If* $(u^\varepsilon, \theta^\varepsilon) \in \mathcal{W}_\varepsilon$ *is the solution of problem* (4.1)-(4.7)*, where* $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ *and* $\theta^\varepsilon = (\theta^{1\varepsilon}, \theta^{2\varepsilon})$*, then we have*

$$\tilde{u}^{1\varepsilon} \overset{*}{\rightharpoonup} |Y_1|\cdot u^1 \quad \textit{weakly* in}\ \ L^\infty(0,T;L^2(\Omega))^N, \tag{4.15}$$

$$\tilde{u}^{2\varepsilon} \overset{*}{\rightharpoonup} |Y_2|\cdot u^1 + f_l\cdot\xi^l+\theta^2\cdot\zeta \ \textit{weakly* in}\ \ L^\infty(0,T;L^2(\Omega))^N, \tag{4.16}$$

$$\tilde{\theta}^{\alpha\varepsilon} \overset{*}{\rightharpoonup} |Y_\alpha|\cdot\theta^\alpha \textit{weakly* in}\ L^\infty(0,T;L^2(\Omega)),\quad \forall\alpha\in\{1,2\}, \tag{4.17}$$

where (u,θ) *with* $u=(u^1,u^2)$ *and* $\theta=(\theta^1,\theta^2)$ *is the uniique solution of problem*

$$-\frac{\partial}{\partial x_j}\Big(a^{1*}_{ijkh}\frac{\partial u^1_k}{\partial x_h}-b^{1*}_{ij}\theta^1\Big)+\langle\rho^1\rangle_{Y_1}\frac{\partial^2u^1_i}{\partial t^2}=f_i\quad \textit{in}\ \Omega, \tag{4.18}$$

$$-\frac{\partial}{\partial x_i}\Big(k^{1*}_{ij}\frac{\partial\theta^1}{\partial x_j}\Big)+T_0b^{1*}_{ij}\frac{\partial e_{ij}(u^1)}{\partial t}+\Big(T_0\gamma^{1*}+\langle c^1\rangle_{Y_1}\Big)\frac{\partial\theta^1}{\partial t}=|Y_1|\,r\quad \textit{in}\ \Omega, \tag{4.19}$$

$$\Big(T_0\gamma^{2*}+\langle c^2\rangle_{Y_2}\Big)\frac{\partial\theta^2}{\partial t}=|Y_2|\,r\quad \textit{in}\ \Omega, \tag{4.20}$$

$$u^1=0,\quad \theta^1=0,\quad \textit{on}\ \partial\Omega, \tag{4.21}$$

with the initial conditions

$$u^\alpha(0,x)=0,\quad \dot{u}^\alpha(0,x)=0,\quad \theta^\alpha(0,x)=0, \tag{4.22}$$

where the components of vector fields ξ^l *and* ζ *are* $\xi^l_i=\int_{Y_2}w^l_{2i}$ *and* $\zeta_i=\int_{Y_2}z^2_i$*, respectively.*

This paper was supported by the Sectorial Operational Programme Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under the contract number SOP HRD/107/1.5/S/82514.

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